Application of mathematical solution in heat conduction with Flexpde
Chapter 1

Introduction to heat transfer and FlexPDE software
• **What is Heat?**

• All matter is made up of molecules and atoms. These atoms are always in different types of motion (translation, rotational, vibrational).

• The motion of atoms and molecules creates heat or thermal energy. All matter has this thermal energy.

• The more motion the atoms or molecules have the more heat or thermal energy they will have.
• What is temperature?

• Temperature is an average value of energy for all the atoms and molecules in a given system.

• Temperature is independent of how much matter there is in the system.

• It is simply an average of the energy in the system.
Since \( T(x + dx) - T(x) = dT \)

the above gives

\[
q_x = kA \frac{dT}{dx}.
\]

It is useful to introduce the term heat flux \( q_x \) which is defined as the heat flow rate per unit surface area normal to \( x \). Thus,

\[
\dot{q}_x = \frac{q_x}{A}.
\]

\[
\dot{q}_x = -k \frac{dT}{dx}, \dot{q}_y = -k \frac{dT}{dy}, \dot{q}_z = -k \frac{dT}{dz}.
\]
• How is heat transferred?

• Heat can travel from one place to another in three ways: Conduction, Convection and Radiation.

• Both conduction and convection require matter to transfer heat.

• If there is a temperature difference between two systems heat will always find a way to transfer from the higher to lower system.
Conduction rate equation is described by the Fourier law.

\[ q_x = k \frac{A(T_{si} - T_{so})}{L}, \]

\[ q_x = k_A \frac{T(x) - T(x + dx)}{L} = -k_A \frac{T(x + dx) - T(x)}{L}, \]
Thermal energy is transferred from hot places to cold places by convection.

Convection occurs when warmer areas of a liquid or gas rise to cooler areas in the liquid or gas. Cooler liquid or gas then takes the place of the warmer areas which have risen higher. This results in a continuous circulation pattern.
Radiation is a method of heat transfer that does not rely upon any contact between the heat source and the heated object as is the case with conduction and convection.

Heat can be transmitted though empty space by thermal radiation often called infrared radiation. This is a type electromagnetic radiation. No mass is exchanged and no medium is required in the process of radiation.

\[ E_b = \sigma T_{abc}^4, \]

\[ \sigma = \text{Stefan Boltzman constant}, \ 5.67 \times 10^{-8} \frac{W}{m^2 \cdot K^4}. \]

\[ T_{abc} = \text{Absolute temperature of the emitting surface, } K. \]
What is FlexPDE?

FlexPDE is a modelling software based on finite element methods which uses codes and is a numerical solver.

- Editing and preparation of text
- Creating finite element mesh
- Finite element solver to find the results
- Graphical results to display the findings
The main sections of the code in this software are:

<table>
<thead>
<tr>
<th>Title</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>TITLE</td>
<td>An explanatory term for the outputs.</td>
</tr>
<tr>
<td>SELECT</td>
<td>Gives the user the ability to change the predefined parameters of FlexPDE.</td>
</tr>
<tr>
<td>VARIABLES</td>
<td>To name the dependant variables.</td>
</tr>
<tr>
<td>DEFINITIONS</td>
<td>To define the parameters, relationships or functions.</td>
</tr>
<tr>
<td>EQUATIONS</td>
<td>Each variable is included in a differential equation or a partial derivative.</td>
</tr>
<tr>
<td>BOUNDARIES</td>
<td>The geometry is created in the boundary of the domain, by putting the linear or arc sections next to each other.</td>
</tr>
<tr>
<td>MONITORS and PLOTS</td>
<td>A desired graphical output which include every combination of CONTOUR, SURFACE, ELEVATION, and VECTOR.</td>
</tr>
<tr>
<td>END</td>
<td>Completes the text</td>
</tr>
</tbody>
</table>
A simple equation of diffusion on a square is as follows:

```
TITLE 'Simple diffusion equation'
{ this problem lacks sources and boundary conditions }
VARIABLES
U
DEFINITIONS
k = 3{conductivity}
DEFINITIONS
Div (k * grad (u)) = 0
BOUNDARIES
REGION 1
Start (0,0)
Line to (1,0)to (1,1)to (0,1) to Close
PLOTS
CONTOUR (u)
Vector (k * grad (u))
End.
```
Way to define boundary conditions

- The boundary conditions of VALUE, defines the value of a variable at the border.

- The boundary conditions of NATURAL, defines the value of flux at the border.

```plaintext
...  
BOUNDARIES  
REGION 1  
Start (0,0)  
VALUE (u) = 0 Line To (1,0) {fixed value on bottom}  
VALUE (u) = 1 Line To (0,1) {fixed value on top}  
NATURAL (u) = 0 Line To Close {insulated left side}  
...  
```
Steps to define the problem in FlexPDE

- Defining variables and equations
- Specifying the geometry of the problem
- Defining the properties of the materials
- Applying boundary conditions
- Specifying the graphical results
Code writing rules in FlexPDE

✓ Differentiating, such as \( \frac{du}{dx} \) is shown as \( dx(u) \)

coordinate systems are recognized reliable, such as second order derivatives \( dxx(u) \)

and differential operators such as \( \text{div, Grad, and curl} \)

✓ The names are not sensitive to lower or upper case letters,

✓ Notes and descriptions can be included in the text. \{ \} is used for notes while exclamation mark is used to ignore the rest of the line.
Variables and equations:

- What variables need to be analyzed?

- Which differential equations with partial derivatives describe these variables?

\[
\begin{align*}
\text{VARIABLES} & : \\
\Phi & , \ 
\text{A}, \ 
\text{B} \\
\text{EQUATIONS} & : \\
\text{Div} \left( \text{grad} \left( \Phi \right) \right) & = 0 \\
\text{A} : \text{Div} \left( \text{grad} \left( \text{A} \right) \right) & = 0 \\
\text{B} : \text{Div} \left( \text{grad} \left( \text{B} \right) \right) & = 0
\end{align*}
\]
Creating the final code by using the previous sections’ code

```plaintext
TITLE 'Heat flow around an Insulating blob'
VARIABLES
    Phi { the temperature }
DEFINITIONS
    k = 1 { default conductivity }
    R = 0.5 { blob radius }
EQUATIONS
    Div (-k * grad(Phi)) = 0
BOUNDARIES
    REGION 1 'box'
    START (-1, -1)
    VALUE (Phi) = 0 LINE TO (1, -1)
    NATURAL (Phi) = 0 LINE TO (1,1)
    VALUE (Phi) = 1 LINE TO (-1,1)
    NATURAL (Phi) = 0 LINE TO CLOSE
    REGION 2 'blob' { the embedded circular 'blob' }
```
\( k = 0.001 \)

\( \text{START 'ring'\( (R, 0) \)} \)

\( \text{ARC (CENTER = 0, 0) ANGLE = 360 TO CLOSE PLOTS} \)

\( \text{CONTOUR (\( \Phi \))} \)

\( \text{VECTOR (} -k \times \text{grad (\( \Phi \)))} \)

\( \text{Elevation (\( \Phi \)) from (0, -1) to (0, 1)} \)

\( \text{Elevation (normal (} -k \times \text{grad (\( \Phi \))) on 'ring' \)} \)

\( \text{End.} \)
The outputs of this text are as follows:
• This does not mean that FlexPDE can guarantee the accuracy of 0.2 percent on the domain.

In the previous problem, it is possible to add the expression as a new section:

\[
SELECT \ Errlim = 1e - 5. 
\]

FlexPDE corrects the network twice and creates a mesh network as such:
Cylindrical geometry

- **XCYLINDER** puts the cylindrical rotation axis in $z-axis$ along $x-axis$, so that the radius is along the vertical direction. The coordinate system is the $(Z,R)$ system.

- **YCYLINDER** puts the cylindrical rotation axis in $z-axis$ along $y-axis$, so that the radius is along the horizontal direction. The coordinate system is the $(R,Z)$ system.
The full text which is transformed to a cylindrical coordinate system is as follows:

```plaintext
TITLE 'Heat flow around an Insulating Torus'
COORDINATES
YCYLINDER
VARIABLES
  Phi { the temperature }
DEFINITIONS
  k =1 { default conductivity }
  R = 0.5 { blob radius }
EQUATIONS
  Div (−k * grad (Phi)) = 0
BOUNDARIES
  REGION 1 'box'
  START (−1,−1)
  VALUE (Phi) = 0 LINE TO (2,−1)
  NATURAL (Phi) = 0 LINE TO (2,1)
  VALUE (Phi) = 1 LINE TO (0,1)
```
NATURAL(Phi) = 0 LINE TO CLOSE
REGION 2 'blob' { the embedded circular 'blob' }
k = 0.001
START 'ring'(R,0)
ARC (CENTER = 0,0) ANGLE = 360 TO CLOSE
PLOTS
CONTOUR (Phi)
VECTOR (−k * grad (Phi))
Elevation (Phi) from (1,−1) to (1,1)
Elevation (normal (−k * grad (Phi))) on 'ring'
End.
The contour and the resulted plot is as follows:
Time-dependant problems

- **THRESHOLD** is the meaningful value for each variable (the value assigned to **THRESHOLD** is the limit of changes that the user neglects any smaller changes).

- Time-dependant *PDE*

- Time limit for solving the problem

- The time that the plot needs to be generated

- The plots considered from the changes of a variable in a specific point along the length of the time (variable’s history)
Now the complete text with the corresponding expression is as follows:

```
TITLE 'Transient Heat flow around an Insulating blob'
VARIABLES
  Phi(threshold = 0.01) { the temperature }
DEFINITIONS
  k = 1 { default conductivity }
  C = 1 { default heat capacity }
  R = 0.5 { blob radius }
EQUATIONS
  Div (−k * grad (Phi)) + C * dt (Phi) = 0
```
BOUNDARIES
REGION 1 'box'
START (-1,-1)
VALUE (\Phi) = 0 LINE TO (1,-1)
NATURAL (\Phi) = 0 LINE TO (1,1)
VALUE (\Phi) = Sin(t) LINE TO (-1,1)
NATURAL (\Phi) = 0 LINE TO CLOSE
REGION 2 'blob'\{ the embedded 'blob'\}
k = 0.001
C = 0.01
START (R,0)
ARC (CENTER = 0,0) ANGLE = 360
TIME 0 TO 2*Pi
PLOTS
FOR T = Pi / 2 by Pi / 2 TO 2*Pi
CONTOUR(Phi)
VECTOR(-k * grad(Phi))
Elevation(Phi) from (0, -1) to (0, 1)
Elevation(normal(-k * grad(Phi))) on 'ring'

HISTORIES
HISTORY(Phi) AT (0, r / 2)(0, r) (0, 3*r / 2)
End.
Analysis of more complex problem

Coefficients and non-linear equations:

```plaintext
REGION 2 'blob' { the embedded 'blob' }
k = esp(-5*Phi)
...```
EQUATIONS

\[ \text{Div} (-k \cdot \text{grad} (\Phi)) + 0.01 \cdot \Phi \cdot \sin(\Phi) = 0 \]

Click on RUN and the way to solve the problem appears:
Utilising FlexPDE for three-dimensional problems

Extrusion

```
EXTRUSION
SURFACE 'Bottom' z = 0
LAYER 'Everything'
SURFACE 'Top' z = 0
```
EXTRUSION
SURFACE "Bottom" z = -1/2
LAYER "Underneath"
SURFACE "Can Bottom" z = -1/4
LAYER "Can"
SURFACE "Can Top" z = 1/4
LAYER "Above"
SURFACE 'Top' z = 0
Materials properties adjustment in regions and layers

REGION 2 'blob' \{ the embedded blob \}
LAYER " Can "
k = 0.001
START 'Ring (R, 0)
ARC ( CENTER = 0,0 ) ANGLE = 360
BOUNDARIES
REGION 1
\textit{Params (1, all ) \{ parameter redefinitions for all layers of region 1\}}
\textit{LAYER 1}
\textit{Params (1,1) \{ parameters redefinitions restricted to layer 1 of region 1 \}}
\textit{LAYER 2}
\textit{Params (1,2) \{ parameters redefinitions restricted to layer 2 of region 1 \}}
\textit{LAYER 3}
\textit{Params (1,3) \{ parameters redefinitions restricted to layer 3 of region 1 \}}
\textit{START ( , ) .... TO CLOSE \{ trace the perimeter \}}
REGION 2
\textit{Params (2, all ) \{ parameter redefinitions for all layers of region 2\}}
\textit{LAYER 1}
\textit{Params (2,1) \{ parameters redefinitions restricted to layer 1 of region 2 \}}
\textit{LAYER 2}
\textit{Params (2,2) \{ parameters redefinitions restricted to layer 2 of region 2 \}}
\textit{LAYER 3}
\textit{Params (2,3) \{ parameters redefinitions restricted to layer 3 of region 2 \}}
\textit{START ( , ) .... TO CLOSE \{ trace the perimeter \}}
\textit{\{ ..... and so forth for all regions \}}
Defining empty areas

REGION 2 'blob'
LAYER 'Can' VOID
START 'ring' (R,0)
ARC (CENTER = 0,0 ) ANGLE = 360
Defining restricted areas

LIMITED REGION 2' blob' { the embedded blob }
LAYER ' Can ' K = 0.001
START ' ring ' (R,0)
ARC ( CENTER = 0,0 ) ANGLE = 360 TO CLOSE
Defining the plots in special pages

```
TITLE 'Heat flow around an Insulating Canister'
COORDINATES
CARTESIAN 3
VARIABLES
Phi  { the temperature }
DEFINITIONS
K =1  { default conductivity }
R =0.5  { blob radius }
EQUATIONS
Div (−k * grad (Phi )) = 0
EXTRUSION
```
SURFACE "Bottom" z = -1/2
LAYER "Underneath"
SURFACE " Can Bottom " z = -1/4
LAYER " Can "
SURFACE " Can Top " z = 1/4
LAYER " Above "
SURFACE 'Top' z = 1/2
BOUNDARIES
REGION 1 'box'
START (-1,-1)
VALUE (Phi) = 0 LINE TO (1,-1)
NATURAL (Phi) = 0 LINE TO (1,1)
VALUE (Phi) = 1 LINE TO (-1,1)
NATURAL (Phi) = 0 LINE TO CLOSE
LIMITED REGION 2 'blob' \{the embedded blob\}

\textbf{LAYER 2} \textit{K} = 0.001 \{ \textit{the canister only} \}

\textbf{START} 'ring' \((R,0)\)

\textbf{ARC} \((\textit{CENTER} = 0,0)\) \textbf{ANGLE} = 360 \textit{TO CLOSE}

\textbf{PLOTS}

\textbf{GRID} \((y,z)\) \textit{on} \(x = 0\)

\textbf{CONTOUR} \((\textit{Phi})\) \textit{on} \(x = 0\)

\textbf{VECTOR} \((-k * \textit{grad}(\textit{Phi})\) \textit{on} \(x = 0\)

\textbf{ELEVATION} \((\textit{Phi})\) \textit{from} \((0,-1,0)\) \textit{To} \((0,1,0)\) \{ \textit{note 3D coordinates} \}

\textbf{End.}
Adjusting the boundary conditions in three-dimensional problems
BOUNDARIES
SURFACE 1
s(all, 1) \{ BC" s on surface 1 over full domain \}
SURFACE 2
s(all, 2) \{ BC" s on surface 2 over full domain \}
{.... Other surfaces }
REGION 1
SURFACE 1
\( s (1,1) \) \{ \text{BC" s on surface 1, restricted to region 1} \}

\text{SURFACE 2}

\( s (1,2) \) \{ \text{BC" s on surface 2, restricted to region 1} \}

....

\text{START (,) \{ ....Begin the perimeter of region m \} }

\( w (1,\ldots) \) \{ \text{BC"s on following segments of sidewall of region 1 on all layers} \}

\text{LAYER 1}

\( w (1,1) \)

\{ \text{BC"s on following segments of sidewall of region 1, restricted to layer 1} \}

\text{LAYER 2}

\( w (1,2) \)

\{ \text{BC"s on following segments of sidewall of region 1, restricted to layer 2} \}

....
LINE TO....
{ segments of the base plane boundary with above BC" s }

LAYER 1
w (1,1)

{new BC" s on following segments of sidewall of region 1, restricted to layer 1}
...

LINE TO....
{ continue the perimeter of region 1 with modified boundary conditions }

TO CLOSE

REGION 2
SURFACE 1
s ( 2, 1) { BC" s on surface 1, restricted to region 2 }

SURFACE 2
\( s \ (2, 2) \) \{ BC " s on surface 2, restricted to region 2 \}

... 
\textit{START} (,) \{ .... Begin the perimeter of region m \}

\( w \ (2,...) \) \{ BC " s on following segments of sidewall of region 2 on all layers \}

\textbf{LAYER 1}

\( w \ (2,1) \)

\{BC " s on following segments of sidewall of region 2, restricted to layer 1\}

\textbf{LAYER 2}

\( w \ (2,2) \)

\{BC " s on following segments of sidewall of region 2, restricted to layer 2\}

... 
\textit{LINE TO}....

\{ segments of the base plane boundary with above BC " s \}

\textbf{LAYER 1}

\( w \ (2,1) \)

\{new BC " s on following segments of sidewall of region 2, restricted to layer 1\}

... 
\textit{LINE TO}....

\{ continue the perimeter of region 2 with modified boundary conditions \}

\textbf{TO CLOSE}
Describing the contact surface of different materials with the special shape

\[ Z = Z_{center} + \sqrt{R^2 - x^2 - y^2} \]

Or

\[ Z = Z_{top} - R + \sqrt{R^2 - x^2 - y^2} \]

TITLE 'Heat flow around Insulating Sphere'

BOUNDARIES
CARTESIAN 3
VARIABLES
Phi \{ the temperature \}
DEFINITIONS
K = 1 \{default conductivity\}
R = 0.5 \{sphere radius\}
\{shape of hemispherical cap:\}
Zsphere = \sqrt{\max\left(R^2 - x^2 - y^2, 0\right)}

EQUATIONS
Div(-k \cdot \text{grad}(\Phi)) = 0

EXTRUSION
SURFACE 'Bottom' \quad z = -1
LAYER 'Underneath'
SURFACE 'Sphere Bottom' \quad z = -\max(Zsphere, 0)
LAYER 'Can'
SURFACE 'Sphere Top' \quad z = -\max(Zsphere, 0)
LAYER 'Above'
SURFACE 'Top' \quad z = 1
BOUNDARIES
REGION 1 'box'
START (-1,-1)
VALUE (Phi) = 0 LINE TO (1,-1)
NATURAL (Phi) = 0 LINE TO (1,1)
VALUE (Phi) = 1 LINE TO (-1,1)
NATURAL (Phi) = 0 LINE TO CLOSE
REGION 2 'blob' {the embedded blob}
LAYER 2 k = 0.001
START 'ring' (Rsphere,0)
ARC (CENTER = 0,0) ANGLE = 360 TO CLOSE
PLOTS
GRID (y, z) on x = 0
CONTOUR (-k * grad (Phi) on x = 0
VECTOR (-k * grad (Phi) on x = 0
ELEVATION (Phi) from (0,-1,0) TO (0,1,0)
End.
**Function of surface generation**

**DEFINITIONS**

\[ R_0 = 1 \{ \text{cylinder radius} \} \]
\[ \text{Len} = 3 \{ \text{cylinder length} \} \]
\[ \theta = 45 \{ \text{axis direction in degrees} \} \]
\[ C = \cos(\theta \text{ degrees}) \{ \text{direction cosines of the axis direction} \} \]
\[ s = \sin(\theta \text{ degrees}) \]
\[ x_0 = -(\text{Len} / 2) \times C \{ \text{beginning point of the cylinder axis} \} \]
\[ y_0 = -(\text{Len} / 2) \times s \]
\[ z_{\text{off}} = 10 \]
\[ Z_s = \text{CYLINDER}((x_0, y_0, 0.5), (x_0 + \text{Len} \times C, y_0 + \text{Len} \times s, 0.5), R_0) \]

**EXTRUSION**

\[ \text{SURFACE } z = z_{\text{off}} - z_s \{ \text{the bottom half - surface} \} \]
\[ \text{SURFACE } z = z_{\text{off}} + z_s \{ \text{the top half - surface} \} \]
SURFACE \( z = z_{\text{off}} - zs \) \{ the bottom half \} \{ surface \}
SURFACE \( z = z_{\text{off}} + zs \) \{ the top half \} \{ surface \}

BOUNDARIES

REGION 1

START \((x_0, y_0)\)

LINE TO \((x_0 + R0*C, y_0 - R0*s)\)

TO \((x_0 + Len*C + Ro*C, y_0 + Len*s - R0*s)\)

TO \((x_0 + Len*C - Ro*C, y_0 + Len*s - R0*s)\)

TO \((x_0 - Ro*C, y_0 + R0*s)\)

TO CLOSE